A Simple Proof of Cohen's Theorem

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Let M be a module over a commutative ring R. Then M is called a *Noetherian module* if every submodule of M is finitely generated, and R is called a *Noetherian ring* if it is a Noetherian module over itself. Cohen proved that a commutative ring R is Noetherian if and only if every prime ideal in R is finitely generated (see, for example, [**1**] or [**3**]). Jothilingam has recently given a generalization of Cohen's theorem for modules:

Theorem. *Let R be a commutative ring and M a finitely generated R-module. Then M is Noetherian if and only if the submodule $\mathfrak{p}M$ is finitely generated for every prime ideal $\mathfrak{p}$ of R.*

By adapting the argument in [**2**], we will give a simple proof for this theorem, one that doesn't require the theory of associated prime ideals. We remind the reader that for an R-module M the set $\{r \in R : rM = 0\}$ is called the *annihilator* of M and is denoted by $\mathrm{Ann}(M)$.

Proof. Suppose that M is not Noetherian. By Zorn's Lemma there exists a proper submodule N of M that is maximal among the nonfinitely generated submodules of M. We first show that $\mathrm{Ann}(M)/N = \mathfrak{p}$ is a prime ideal. Suppose that ab belongs to $\mathfrak{p}$, but that neither a nor b is in $\mathfrak{p}$. Then $N + aM$ and $N + bM$ are both finitely generated. Assume that $\{n_i + am_i\}_{i=1}^{\ell}$ is a set of generators $N + aM$, where n_i is in N and m_i in M. Put $L = \{m \in M : am \in N\}$. It is easy to see that L is a submodule of M containing both N and bM. By the maximality of N, L is finitely generated. We show that

$$N = \sum_{i=1}^{\ell} Rn_i + aL.$$

Consider y in N. Since y belongs to $N + aM$, there exist $b_1, \ldots, b_\ell$ in R such that

$$y = \sum_{i=1}^{\ell} b_i(n_i + am_i) = \sum_{i=1}^{\ell} b_i n_i + a \sum_{i=1}^{\ell} b_i m_i.$$

This means that $a \sum_{i=1}^{\ell} b_i m_i$ lies in N, whence y is a member of the ideal

$$\sum_{i=1}^{\ell} Rn_i + aL.$$

Since the other inclusion is trivial, we get $N = \sum_{i=1}^{\ell} Rn_i + aL$. It follows that N is finitely generated, which contradicts the definition of N. Therefore $\mathfrak{p}$ is a prime ideal.

Since M is finitely generated, we have $M/N = R\overline{x_1} + \cdots + R\overline{x_t}$ for some $x_1, \ldots, x_t$ in M, where $\overline{x}$ signifies the equivalence class of x in M/N, hence $\mathfrak{p} = \cap_{i=1}^{t} \text{Ann}(R\overline{x_i})$. Because $\mathfrak{p}$ is a prime ideal, $\mathfrak{p} = \text{Ann}(R\overline{x_j})$ for some j. Suppose that the set $\{y_i + r_i x_j\}_{i=1}^{k}$ generates $N + Rx_j$, where y_i is in N and r_i in R. By an argument similar to the earlier one, we have $N = \sum_{i=1}^{k} Ry_i + \mathfrak{p}x_j$. Since $\mathfrak{p}M$ is contained in N, we obtain

$$N = \sum_{i=1}^{k} Ry_i + \mathfrak{p}x_j \subseteq \sum_{i=1}^{k} Ry_i + \mathfrak{p}M \subseteq \sum_{i=1}^{k} Ry_i + N \subseteq N.$$

It follows that $N = \sum_{i=1}^{k} Ry_i + \mathfrak{p}M$ is a finitely generated submodule of M, a contradiction to the choice of N. Thus M is a Noetherian module. The converse is clear. ∎

REFERENCES

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