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Author(s): A. R. Naghipour
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/30037605
Accessed: 08/03/2010 10:40

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A Simple Proof of Cohen’s Theorem

A. R. Naghipour

Let $M$ be a module over a commutative ring $R$. Then $M$ is called a Noetherian module if every submodule of $M$ is finitely generated, and $R$ is called a Noetherian ring if it is a Noetherian module over itself. Cohen proved that a commutative ring $R$ is Noetherian if and only if every prime ideal in $R$ is finitely generated (see, for example, [1] or [3]). Jothilingam has recently given a generalization of Cohen’s theorem for modules:

**Theorem.** Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. Then $M$ is Noetherian if and only if the submodule $pM$ is finitely generated for every prime ideal $p$ of $R$.

By adapting the argument in [2], we will give a simple proof for this theorem, one that doesn’t require the theory of associated prime ideals. We remind the reader that for an $R$-module $M$ the set $\{r \in R : rM = 0\}$ is called the annihilator of $M$ and is denoted by $\text{Ann}(M)$.

**Proof.** Suppose that $M$ is not Noetherian. By Zorn’s Lemma there exists a proper submodule $N$ of $M$ that is maximal among the nonfinitely generated submodules of $M$. We first show that $\text{Ann}(M)/N = p$ is a prime ideal. Suppose that $ab$ belongs to $p$, but that neither $a$ nor $b$ is in $p$. Then $N + aM$ and $N + bM$ are both finitely generated. Assume that $\{n_i + am_i\}_{i=1}^\ell$ is a set of generators $N + aM$, where $n_i$ is in $N$ and $m_i$ in $M$. Put $L = \{m \in M : am \in N\}$. It is easy to see that $L$ is a submodule of $M$ containing both $N$ and $bM$. By the maximality of $N$, $L$ is finitely generated. We show that

$$N = \sum_{i=1}^\ell Rn_i + aL.$$
Consider \( y \in N \). Since \( y \) belongs to \( N + aM \), there exist \( b_1, \ldots, b_\ell \) in \( R \) such that
\[
y = \sum_{i=1}^{\ell} b_i(n_i + am_i) = \sum_{i=1}^{\ell} b_in_i + a\sum_{i=1}^{\ell} b_im_i.
\]
This means that \( a\sum_{i=1}^{\ell} b_im_i \) lies in \( N \), whence \( y \) is a member of the ideal
\[
\sum_{i=1}^{\ell} Rn_i + aL.
\]
Since the other inclusion is trivial, we get \( N = \sum_{i=1}^{\ell} Rn_i + aL \). It follows that \( N \) is finitely generated, which contradicts the definition of \( N \). Therefore \( p \) is a prime ideal.

Since \( M \) is finitely generated, we have \( M/N = R\overline{x_1} + \cdots + R\overline{x_t} \) for some \( x_1, \ldots, x_t \) in \( M \), where \( \overline{x} \) signifies the equivalence class of \( x \) in \( M/N \), hence \( p = \bigcap_{j=1}^{t} \text{Ann}(R\overline{x_j}) \). Because \( p \) is a prime ideal, \( p = \text{Ann}(R\overline{x_j}) \) for some \( j \). Suppose that the set \( \{y_i + r_ix_j\}_{i=1}^{k} \) generates \( N + Rx_j \), where \( y_i \) is in \( N \) and \( r_i \) in \( R \). By an argument similar to the earlier one, we have \( N = \sum_{i=1}^{k} Ry_i + px_j \). Since \( pM \) is contained in \( N \), we obtain
\[
N = \sum_{i=1}^{k} Ry_i + px_j \subseteq \sum_{i=1}^{k} Ry_i + pM \subseteq \sum_{i=1}^{k} Ry_i + N \subseteq N.
\]
It follows that \( N = \sum_{i=1}^{k} Ry_i + pM \) is a finitely generated submodule of \( M \), a contradiction to the choice of \( N \). Thus \( M \) is a Noetherian module. The converse is clear.

REFERENCES


Department of Mathematics, Shahre-Kord University, P.O. Box: 115, Shahre-Kord, IRAN
arnaghip@ipm.ac.ir