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A Simple Proof of Cohen's Theorem

A. R. Naghipour

Let M be a module over a commutative ring R . Then M is called a *Noetherian module* if every submodule of M is finitely generated, and R is called a *Noetherian ring* if it is a Noetherian module over itself. Cohen proved that a commutative ring R is Noetherian if and only if every prime ideal in R is finitely generated (see, for example, [1] or [3]). Jothilingam has recently given a generalization of Cohen's theorem for modules:

Theorem. *Let R be a commutative ring and M a finitely generated R -module. Then M is Noetherian if and only if the submodule $\mathfrak{p}M$ is finitely generated for every prime ideal \mathfrak{p} of R .*

By adapting the argument in [2], we will give a simple proof for this theorem, one that doesn't require the theory of associated prime ideals. We remind the reader that for an R -module M the set $\{r \in R : rM = 0\}$ is called the *annihilator* of M and is denoted by $\text{Ann}(M)$.

Proof. Suppose that M is not Noetherian. By Zorn's Lemma there exists a proper submodule N of M that is maximal among the nonfinitely generated submodules of M . We first show that $\text{Ann}(M)/N = \mathfrak{p}$ is a prime ideal. Suppose that ab belongs to \mathfrak{p} , but that neither a nor b is in \mathfrak{p} . Then $N + aM$ and $N + bM$ are both finitely generated. Assume that $\{n_i + am_i\}_{i=1}^{\ell}$ is a set of generators $N + aM$, where n_i is in N and m_i in M . Put $L = \{m \in M : am \in N\}$. It is easy to see that L is a submodule of M containing both N and bM . By the maximality of N , L is finitely generated. We show that

$$N = \sum_{i=1}^{\ell} Rn_i + aL.$$

Consider y in N . Since y belongs to $N + aM$, there exist b_1, \dots, b_ℓ in R such that

$$y = \sum_{i=1}^{\ell} b_i(n_i + am_i) = \sum_{i=1}^{\ell} b_in_i + a \sum_{i=1}^{\ell} b_im_i.$$

This means that $a \sum_{i=1}^{\ell} b_im_i$ lies in N , whence y is a member of the ideal

$$\sum_{i=1}^{\ell} Rn_i + aL.$$

Since the other inclusion is trivial, we get $N = \sum_{i=1}^{\ell} Rn_i + aL$. It follows that N is finitely generated, which contradicts the definition of N . Therefore \mathfrak{p} is a prime ideal.

Since M is finitely generated, we have $M/N = R\bar{x}_1 + \dots + R\bar{x}_t$ for some x_1, \dots, x_t in M , where \bar{x} signifies the equivalence class of x in M/N , hence $\mathfrak{p} = \bigcap_{i=1}^t \text{Ann}(R\bar{x}_i)$. Because \mathfrak{p} is a prime ideal, $\mathfrak{p} = \text{Ann}(R\bar{x}_j)$ for some j . Suppose that the set $\{y_i + r_ix_j\}_{i=1}^k$ generates $N + Rx_j$, where y_i is in N and r_i in R . By an argument similar to the earlier one, we have $N = \sum_{i=1}^k Ry_i + \mathfrak{p}x_j$. Since $\mathfrak{p}M$ is contained in N , we obtain

$$N = \sum_{i=1}^k Ry_i + \mathfrak{p}x_j \subseteq \sum_{i=1}^k Ry_i + \mathfrak{p}M \subseteq \sum_{i=1}^k Ry_i + N \subseteq N.$$

It follows that $N = \sum_{i=1}^k Ry_i + \mathfrak{p}M$ is a finitely generated submodule of M , a contradiction to the choice of N . Thus M is a Noetherian module. The converse is clear. ■

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